# Grade 6 Math Circles
## February 13 & 14 & 15, 2024
## Knights and Knaves

## Boolean Algebra

In this Math Circles session, we will solve many logic puzzles in which statements can be either **true** or **false**.

The mathematical study of statements that can be either **true** or **false** is known as Boolean algebra.

Just like in regular algebra (which you may be learning), variables are used to represent facts or objects in the world. Whereas the values of variables in regular algebra can be any number at all, the variables in Boolean algebra can only be either **true** (T) or **false** (F).

**Example 1**

Determine which of the following statements can be represented by a Boolean variable:

1. The number of hairs on my head.
2. The number of hairs on my head is greater than 100,000.
3. John is not lying.
4. The height of the CN Tower.
5. $3 > 4$

*Solution:* Statements 2, 3, and 5 can be represented by a Boolean variable.

**Example 2**

Let $A$ be the statement "I am three years old."
Let $B$ be the statement "I have two hands and two eyes."

What are the values of $A$ and $B$ for you, the reader?

*Solution:* For most readers, $A$ is false and $B$ is true.

### Negation

Negation is the operation of taking the opposite truth value. If a Boolean variable is false, then its corresponding opposite statement is true (and vice versa).

**Example 3**

Let $C$ be the statement that Caleb is more than 2 years old. Suppose that $C$ is false. What can we conclude about Caleb?

*Solution:* We can conclude that Caleb is *not* more than 2 years old. To rephrase this into a statement that doesn't include the word *not*: Caleb is 2 years old or younger.

**Example 4**

Suppose that a fictional island consists only of Knights and Knaves. If an inhabitant of the island is *not* a Knight, what can we then conclude?

*Solution:* Each inhabitant of the island is either a Knight or a Knave. Therefore, we can conclude that an inhabitant that is *not* a Knight is a Knave.

## Knights and Knaves

On a fictional island, there are only two types of inhabitants: Knights and Knaves.

- A Knight always tells the truth.
- A Knave always lies.

**Puzzle 5**

Suppose that there are two people on the island, Alex and Blake, each of whom are either Knights or Knaves. Alex says, "We are both Knaves". Who is a Knight and who is a Knave?

First, we will define some variables. In this case, we

- Let $A$ represent the statement "Alex is a Knight"
- Let $B$ represent the statement "Blake is a Knight"

Each of the variables $A$ and $B$ can be either true or false. The puzzle is to figure out which of the variables are true and which are false.

To solve this problem, we can create a *truth table*. What are the possible combinations of values (true or false) for $A$ and $B$?

| $A$ | $B$ |
|---|---|
| | |

Table 1: Our partial truth table for Puzzle 1 (unfilled).

For each combination of values for $A$ and $B$, we can determine if the claim made by Alex ("We are both Knaves") is true and put this in the tab. Our finished table looks like this:

| $A$ | $B$ | Alex and Blake are both Knaves |
|---|---|---|
| T | T | F |
| T | F | T |
| F | T | F |
| F | F | T |

Table 2: Our completed truth table for Puzzle 1.

**Stop and Think.** How can we use the truth table above to solve the puzzle?

### If and only if

To make use of the truth table, we will first need to learn about the phrase *if and only if*. In logic, the phrase *if and only if* means that two statements are either both true or both false. Let's see an example of this in the puzzle. Recall that statement $A$ is "Alex is a Knight." Also, let $C$ represent Alex's statement ("Alex and Blake are both Knaves").

1. If Alex is a Knight (i.e., statement $A$ is true), then the statement that he makes is a true statement (i.e., statement $C$ is true).
2. If Alex is not a Knight (i.e., he is a Knave, so statement $A$ is false), then the statement that he makes is a false statement (i.e., statement $C$ is false).

So either $\boxed{A \text{ and } C \text{ are both true, or } A \text{ and } C \text{ are both false}}$.

We can use the symbols $\iff$ or $\equiv$ to show that two statements are either both true or both false. For example,

$$A \iff C \quad \text{read as "}A\text{ if and only if }C\text{"}$$
$$A \equiv C \quad \text{read as "}A\text{ is logically equivalent to }C\text{"}$$

**Stop and Think.** Which row in the truth table shows that $A$ and $C$ are both true or both false?

This is the row in which the first and last columns match. We can quickly check that there is only one row like this—the third row.

By reading from this row, we can find the solution to our puzzle. We see that $A = \text{F}$ and $B = \text{T}$. Do you remember what $A$ and $B$ represents? If $A$ is false and $B$ is true, then Alex is a Knave and Blake is a Knight.

| A | B | Alex and Blake are both Knaves |
|---|---|---|
| T | T | F |
| T | F | F |
| F | T | F |
| F | F | T |

Table 3: Our truth table for Puzzle 1. The third row (highlighted) is the only row with matching values between columns 1 and 3.

Let's try a few puzzles on your own. Use a truth table in your answer for each puzzle.

**Puzzle 6**

You meet two people, Abby and Baylor, each of whom is either a Knight or a Knave. Abby says, "Either I am a Knave or Baylor is a Knight (or both)."

What are Abby and Baylor? Create a truth table.

**Puzzle 6 solution**

In our solution, we will

- Let $A$ represent the statement "Abby is a Knight"
- Let $B$ represent the statement "Baylor is a Knight"

An incomplete truth table for this puzzle is:

| $A$ | $B$ | Either Abby is a Knave or Baylor is a Knight |
|---|---|---|
| T | T | |
| T | F | |
| F | T | |
| F | F | |

By considering each of the cases, we can fill in the last column as follows:

| $A$ | $B$ | Either Abby is a Knave or Baylor is a Knight |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

Let $C$ represent the statement "Either Abby is a Knight or Baylor is a Knave." Then we know that $A \iff C$, because the statement that Abby makes is true if and only if Abby is a Knight.

In other words, the entry in the first column must match the entry in the last column in the correct solution. The first row in our truth table is the only row where these columns match.

Therefore, Abby and Baylor are both Knights.

**Puzzle 7**

Two people, Ajay and Blaise, are inhabitants of an island of only Knights and Knaves. Ajay says, "At least one of us is a Knave."

What are Ajay and Blaise? Create a truth table.

**Puzzle 7 solution**

In our solution, we will

- Let $A$ represent the statement "Ajay is a Knight."
- Let $B$ represent the statement "Blaise is a Knight."
- Let $C$ represent the statement "At least one of Ajay and Blaise is a Knave."

We know that $A \iff C$. A truth table for this puzzle is shown below. The only row in which the value of $A$ and $C$ matches is highlighted.

| $A$ | $B$ | At least one of Ajay and Blaise is a Knave |
|---|---|---|
| T | T | F |
| T | F | T |
| F | T | T |
| F | F | T |

Therefore, Ajay is a Knight and Blaise is a Knave.

**Puzzle 8**

You meet two people, Avery and Blair, each of whom is either a Knight or a Knave. Two people are said to be of the same kind if they are both Knights or both Knaves. Avery and Blair make the following statements:

Avery: We are of the same kind.
Blair: We are of different kinds.

What are Avery and Blair? Use a truth table.

**Puzzle 8 solution**

In our solution, we will

- Let $A$ represent the statement "Avery is a Knight."
- Let $B$ represent the statement "Blair is a Knight."
- Let $C$ represent the statement "Avery and Blair are of the same kind."
- Let $D$ represent the statement "Avery and Blair are of different kinds."

Since Avery makes statement $C$ and Blair makes statement $D$, then $A \iff C$ and $B \iff D$. A truth table for this puzzle is shown below.

| $A$ | $B$ | Avery and Blair are of the same kind | Avery and Blair are of different kinds |
|---|---|---|---|
| T | T | T | F |
| T | F | F | T |
| F | T | F | T |
| F | F | T | F |

In the above table, we see that rows 1 and 3 are both consistent with $A \iff C$ and rows 3 and 4 are both consistent with $B \iff D$. However, row 3 is the only row in which both $A \iff C$ and $B \iff D$ hold.

Therefore, Avery is a Knave and Blair is a Knight.

## Impossible situations

What if I told you the following?

**Puzzle 9**

I'm thinking of a number between 1 and 9 that satisfy the following:

- If you add 2 to the number, you'd get 6
- If you multiply the number by 4, you would get 8.

We can attempt to solve the problem using algebra. Let $x$ represent the number that I am thinking of. Then, the following two equations hold:

- x + 2 = 6
- 4x = 8

The first equation is true if and only if $x = 4$ because 4 is the only number we can substitute for $x$ that makes the equation true. Similarly, the second equation is true if and only if $x = 2$. Using the $\iff$ symbol, we can write

$$x + 2 = 6 \iff x = 4$$
$$4x = 8 \iff x = 2.$$

Since both equations need to be true at the same time, then this situation is impossible. The variable $x$ cannot both be 4 and 2 at the same time. Either I am lying about the number that I am thinking about, or I am mistaken in the clues that I am giving.

A simpler example of an impossible situation is as follows.

**Example 10**

There are two sisters, Luna and Delilah. Luna is taller than Delilah. Delilah is also taller than Luna. Should you believe me? Can Luna and Delilah actually exist?

*Solution:* No. It is not possible for two people to both be taller than each other, and these assertions contradict each other.

Here's an example of an impossible situation that may not appear that way at first.

**Example 11**

An irresistible cannonball is a cannonball that knocks over everything in its way. An immovable post cannot be knocked over by anything. What happens when an irresistible cannonball hits an immovable post?

*Solution:* An irresistible cannonball and an immovable post could exist by themselves, but they cannot both exist. If an irresistible cannonball existed, then it would knock down anything in its way, so there couldn't be an immovable post. If instead an immovable post existed, then no cannonball could knock it down; there cannot exist an irresistible cannonball.

Let's look at an impossible situation in the form of a Knights and Knaves puzzle.

**Puzzle 12**

On an island of Knights and Knaves, two inhabitants make the following statements:

Adriel: Baloo is a Knave.
Baloo: Adriel is a Knight.

What are Adriel and Baloo?

**Puzzle 12 solution**

Variable definitions:

- Let $A$ represent the statement "Adriel is a Knight"
- Let $B$ represent the statement "Baloo is a Knave"

We can construct the following truth table.

| $A$ | $B$ | Baloo is a Knave | Adriel is a Knight |
|---|---|---|---|
| T | T | F | T |
| T | F | T | T |
| F | T | F | F |
| F | F | T | F |

Note: The third column is the negation (opposite) of the second column while the fourth column is the same as the first column.

We expect the statement "Baloo is a Knave" to have the same truth value as $A$ and the statement "Adriel is a Knight" to have the same truth value as $B$ since it is Adriel and Baloo who make these statements respectively.

But in the truth table, none of the rows have matching values in the first and third column *and* matching values in the second and fourth column. Therefore, the situation is logically impossible.